Transcribing an Animation: The case of the Riemann Sums

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Abstract

In this paper I present a theoretical analysis (genetic decomposition) of the cognitive constructions for the concept of infinite Riemann sums following Piaget's model of epistemology. This genetic decomposition is primarily based on my own mathematical knowledge as well as on my continual observations of students in the process of learning. Based on this analysis I plan to suggest instructional procedures that motivate the mental activities described in the proposed genetic decomposition. In a later study, I plan to present empirical data in the form of informal interviews with students at different stages of learning. The analysis of those interviews may suggest a review of my initial genetic decomposition.

Introduction

I start by situating the study within my teaching experience. It is often the case that Calculus instructors do not emphasize the idea of Riemann sums, at least not symbolically as a limit of an infinite sum of rectangle areas because the justification of the “formula” involves many “unpopular” concepts. Although this section is covered just about as lightly as the delta-epsilon definition of a limit, yet the visual dynamic process that illustrates the area under the graph of a function as a limit of the area of a sequence of rectangles appeals to students’ senses in general; they are quickly visually persuaded that the error decreases as the number of rectangles increases. An animation of the process can be supported by a simple graphical tool. Instructors face a difficulty in teaching the “transcribing” of this animation using algebraic symbols, to end up with a messy compact limit of an infinite sum. The lengthy ambiguous expression involves finding a limit at infinity, choosing an adequate parameter, lifting appropriate points on the interval, as well as other factors that will be detailed later. To circumvent this clutter of obscurity, many instructors are satisfied with a demo of the animation and putting forward the formula with little justification, if at all, applying it to a few simple cases, and hastily moving to the technicalities of the definite integrals. In this study, I discuss what it takes to mend those links between the visual and the symbolic representation of the process involved.

Framework for research

In my study I use as framework for reserach an interpretation of constructivism and Piaget’s ideas on reflective abstraction (Dubinsky, 1991). According to Piaget mathematical knowledge consists of an interconnected collection of cognitive structures corresponding to individual mathematical concepts; and understanding a concept reduces to constructing one or more schema for it (Dubinsky, 1989). In order for learning to occur, Piaget considers that the student must become aware of some disequilibration and then try to re-equilibrate by reconstructing his or her own mathematical schemas. By equilibration he refers to a process by which a knower attempts to understand an item of information by situating it in his or her overall cognitive system. Such situating occurs as the knower constructs an understanding of the item through a process called reflective abstraction (Dubinsky & Lewin, 1986), which can be described as the general coordination between the different cognitive structures in order to construct more advanced structures. The cognitive tools involved are coordination, interiorization, generalization, encapsulation de-encapsulation, and reversal (Asiala et al., 1996). The purpose from the theoretical analysis is to propose a model of cognition for the concepts involved, referred to as a *genetic decomposition*, a description of how a student might make the constructions that would lead to an understanding of the concepts in question. Genetic decomposition is hypothesized from the researcher’s own mathematical knowledge, but most importantly from observations and interviews with students in the process of learning. According to this theory (Asiala et al., 1996), mental constructions in the genetic decomposition are Action, Process, Object and Schema (hence APOS theory). An *action* conception of a mathematical idea is a description of an understanding that is limited to performing action on that concept. When an action is repeated and reflected upon, it may be interiorized into a *process*. A learner has a *process conception* of a concept when his or her depth of understanding is limited to thinking about the idea as a process without being able to execute an action on it. A process is said to be encapsulated into an *object* when the individual can perform action on it, and decompose it back to the matter from which it was initially formed. Finally a *schema* of a piece of mathematics is an individual's collection of action, processes, objects, and other schemas, linked in a coherent framework in the person's mind. (MacDonald et al., 2000). This paradigm has been applied to diverse topics including functions, mathematical induction, calculus, quantification, and abstract algebra, and equivalence classes and partitions (Hamdan, 2006) has lead to major curriculum changes.

Main Questions directing the research

During my observations I tried to focus on answering the following research questions:

1. Can students visualize the rectangles under the curve?
2. Are students persuaded that the error decreases as the number of rectangles increases? And consequently that there is no error when the number of rectangles grows indefinitely?
3. Can they choose the subintervals using appropriate indexing?
4. Also (later) are they aware, once the number of the rectangles grows indefinitely, of the irrelevance of the choice of those points on the subintervals that determine the height of those rectangles, as well as the uniformity of those subintervals?

Prerequisite schema

The objects or schemas that we consider a student needs to have developed prior to the introduction of the Riemann sums are:

1. A schema for functions of one variable.
2. A schema for real numbers where the objects are the numbers and the processes are the arithmetic and algebraic transformations on numbers (Trigueros et al, 2007).
3. A schema for the Cartesian planes including points as objects, and distances between points as processes and actions.
4. A schema for limits in general including limits at infinity.
5. Actions related to the sigma notation for the finite case. This schema should include basic formulas such as the sum of the first n integers and the sum of their squares.
6. A schema for basic geometry that includes areas of rectangles.
7. A schema for (finite) sequences including observing a pattern of objects and labeling them using appropriate indices. Observing similarities in a pattern and being able to recognize the *i*-th entry as a typical entry in the list.

(Conjectured) genetic decomposition

In what follows I list the stages of the proposed genetic decomposition:

I. First coordinate between the schemas for the real numbers with the schema for the Cartesian plane (including intervals and distances) through the action of subdividing the interval $[a,b]$ into n subintervals, for a fixed positive integer n. Then interiorize these actions into a process that gives all the equidistant n points x_i on that interval $[a,b]$, or just as well that gives the n subintervals $[x_i,x_{i+1}]$ of equal length, $d_n=(b-a)/n$. It will be agreed that a and b will be assigned the points x_0 and x_n, respectively.

II. Then coordinate between the schemas of functions of one variable and the schema of Cartesian plane through the action of evaluating the function f at those stops/spots x_i and then "lifting" those verticals of length $f(x_i)$ from $(x_i, 0)$ to $(x_i , f(x_i))$, on the graph of f. These actions then get interiorized into the process that results in an arrangement of adjacent evenly spaced vertical segments. This is followed by the action of connecting the tops of those verticals using horizontal segments from the point $(x_i, f(x_i))$, say, to the point $(x_{i-1},f(x_i))$, starting at x_1. This last action will be interiorized into the process that results in those adjacent rectangles of equal width but different lengths. These n rectangles are labeled $R_1, R_2, .. R_n$, using the previously mentioned indices.

Now the geometric set up is prepared, and the areas of the resulting rectangles can now be evaluated.

III. Next, coordinate between the schema for basic geometry and that of sequences together with the schema for the sigma notation through the action of evaluating the area ($f(x_i)*d_n$) of one "typical" rectangle (the i_th) from the finite sequence obtained in the previous step. This is followed by the coordination with the schema for sigma notation through the action of summing over all $i=1,\ldots,n$ to obtain the finite sum $\sum f(x_i), dn$. This action is interiorized into a process that results into viewing that sum $\sum f(x_i), d_n$ as a function $S(n)$ of n.

Note that it is quite difficult for students at this stage to foresee that neither x nor i would figure in the last result.

IV. Next coordinate between the schema for limits and the schema for sigma notation through the action of evaluating the limit of $S(n)$ as n tends to infinity.

Comments on the genetic decomposition

Before officially conducting the planned interviews that will eventually produce empirical data, and following the setting of the genetic decomposition, I taught the course once more and was able to make a few observations, reflecting on the proposed cognitive model.

These findings were done informally by making close undocumented in-class observations.

I. Action level formula application

Some students seemed to work at an action level, memorizing some facts and trying to use them (Sfard, 1991). Once the formula is available, they know how to manipulate it by substituting values, but they are far from understanding it, or explaining it, let alone producing it themselves. In particular, they want the formula to produce an output in the form of a numerical result. They do not show evidence of having interiorized the mentioned mathematical objects in the genetic decomposition or having constructed relations between them and their intuitive knowledge of the process.

II. Is it a parameter, a variable or an unknown?

Dealing with parameters as indices in the sum was one of the insurmountable difficulties that were not predicted in the original genetic decomposition**.** The fact that it seems to be a common difficulty for most of the students points out that it needs further study. Also the large number of referents in the formula, n or $(b-a)/n$, i confuses students. It takes the development of a new layer of mathematical generality to believe that an object exists even though it is not numerically determined (Bardini, 2005). The problem boils down to accepting "indeterminacy". So, what is n? Is it an unknown, a variable (Drijvers 2001)? Is it the number of rectangles, or the width of the rectangle, or the position of that rectangle in question? Does it designate the position of the rectangle? Students feel they need to attribute a numerical value to n or i in order to progress in this activity. They want to associate the figure with a number. Some students think of it as a temporary fixed value, a placeholder which is changing, but like the variables, operations can be carried on it the same way as an unknown or a variable. There is a lot of research around the meaning of parameters and how students perceive of this genuine conceptual object that can only be referred to through signs (Bardini, 2005). This algebraic object encloses a unique paradoxical nature since it is fixed, yet it remains indeterminate in that it is not subjected to an inquisitorial procedure that would reveal its hidden numeric identity (as is the case with unknowns). Drijvers (Drijvers, 2001) refers to it as generalizer, an implicit that is conceptually more difficult than a variable, it is rather a *generic* organizer uses mostly in the description of a process.

III. Language used in transcribing

Some students' difficulties have their origin in Language. For example, one difficulty found involved the use of the word "lift" while determining the heights of those rectangles. It seems that this difficulty arises because the students assign a different meaning to the action invoked by the word "lift" from that intended. Building the necessary connections includes paying more attention to the use of language making the meaning of words explicit by means of performing the actions and reflecting and discussing the results.

IV. Separation Visual/Algebraic

Many students don't associate their visualization to the algebraic interpretation (Zazkis et al 1996) as if they separate them totally in their head. It became clearer to me that all students are convinced that the process undoubtedly generates the *exact* area, eventually. However, transcribing the animation into an accurate algebraic/symbolic compact expression is the major difficulty: in Piaget's terms, the description of a numeric schema allows students to obtain a formula by translating it into symbols. However, when translating the worded schema into an algebraic formula, students in general come up with answers that reflect some lack of precision in the meaning that they gave symbols. Their translation of the worded schema suggests that they do not interpret the letters "*n*" as standing for an unspecified figure (despite the fact that I tried to suggest it). Both, *i* and *x* are present in the same line, let alone *a* and *b*. As a broad closing, and in a way as to delay the agony of transcribing the process, it is always good to delay the formalization of the process, by exhibiting many diagrams and demos for the procedure that produces those rectangles and areas.

Conclusion To many, teaching implies a didactical transposition: a display of neatly presented material (Arcavi, 2003); which means the transformation of knowledge and adapting it from its scientific character to the knowledge as it should be taught. This explains how the students or receivers fail to see the way it was discovered which could be visual. All this is done at the expense of neatness of representation. This process (it is claimed) by its very nature, linearizes, compartmentalizes and possibly algorithmetizes knowledge, stripping it from its rich interconnections. As such, many teachers may feel that analytic representations which are sequential in nature seem to be more pedagogically appropriate and efficient (Arcavi, 2003).

In a later study I will conduct a formal quantified research where I plan to choose an initial group of students to participate in the first stage of the study. The students will be chosen from a group of undergraduate students at a private university who had taken the equivalent of a Calculus course in the previous semester. I plan to choose a good, an average, and a weak student to be interviewed. The results obtained will be independently analyzed. On the basis of the results obtained from these interviews, I plan to compare students' constructions of the Riemann sum to the constructions predicted (here) by the genetic decomposition. Based on those findings I will design new instruments for teaching that I will assess again. I plan to measure the level of their coordination predicted by the genetic decomposition that they seem to have constructed.

The good news is that it is unusual that in this case visualization is not a problem in this case and that difficulty didn't seem to inhibit the subsequent understanding of basic mathematical constructs.

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